What is a cone?

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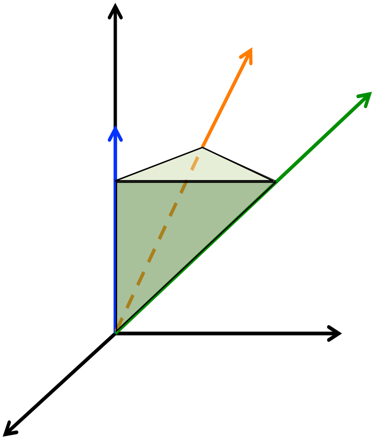
Field of Dreams Conference 2018

Roadmap for today



Intuitive idea of a Cone

"Set of vectors closed under positive combinations"

Intuitive idea of a Cone

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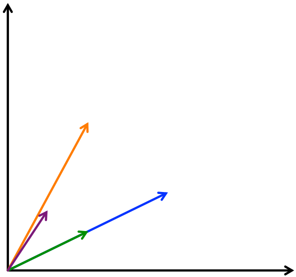
Example

For $V = \left\{ \left(1, \frac{1}{2}\right), (1,2), (2,1), \left(\frac{1}{2}, \frac{3}{4}\right) \right\}$



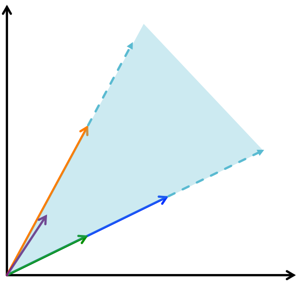
Intuitive idea of a Cone

"Set of vectors closed under positive combinations"

Example

For $V = \left\{ \left(1, \frac{1}{2}\right), (1,2), (2,1), \left(\frac{1}{2}, \frac{3}{4}\right) \right\}$, the cone of V is

$$\mathcal{C}(V) = \left\{ a_1 \left(1, \frac{1}{2}\right) + a_2(1,2) + a_3(2,1) + a_4 \left(\frac{1}{2}, \frac{3}{4}\right) \mid a_i \in \mathbb{R}_{\geq 0} \right\}$$



Vertex/Ray Description

"The space generated by a finite set of vertices/rays"

- Let $V = \{v_1, v_2, \ldots, v_i, r_{i+1}, \ldots, r_m\}$ be a set of vertices and rays in $\mathbb{R}^n$.
- The cone generated by V is

$$\mathcal{C}(V) = \{\lambda_1 v_1 + \cdots + \lambda_m r_m \mid \lambda_i \in \mathbb{R}^n_{\geq 0}\}.$$

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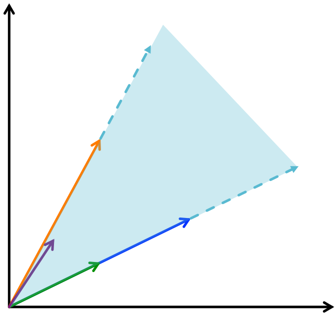
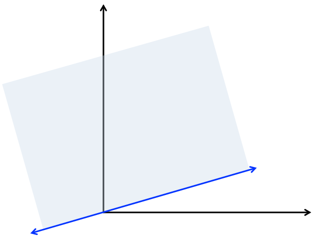


Hyperplane Description

"The intersection of halfspaces"



$$\mathcal{H}_1 = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid \frac{1}{2}x_1 - x_2 \leq 0 \right\}$$

Definition

- A *hyperplane* H is the set $\{x \in \mathbb{R}^n | a(x) = 0\}$, for linear map a over $\mathbb{R}^n$.
- A *closed halfspace* $\mathcal{H}$ is choosing a "side" of H:

$$\{x \in \mathbb{R}^n | a(x) \geq 0\}.$$

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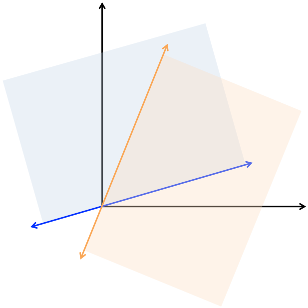

$$\mathcal{H}_1 = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid \frac{1}{2}x_1 - x_2 \leq 0 \right\}$$

$$\mathcal{H}_2 = \{ (x_1, x_2) \in \mathbb{R}^2 \mid 2x_1 - x_2 \geq 0 \}$$

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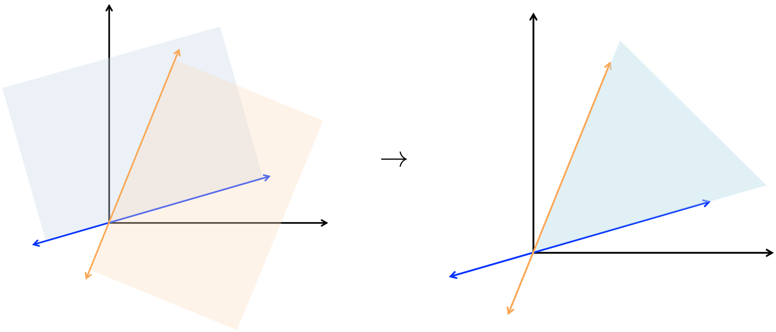
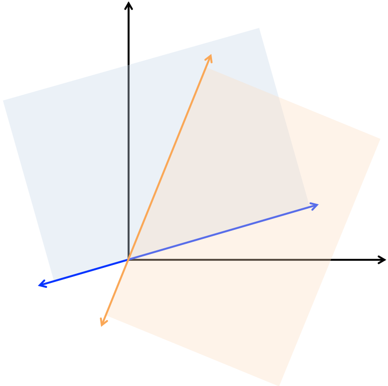
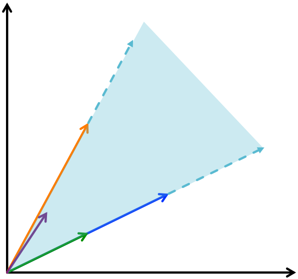


Definition

- A *convex cone* $\mathcal{C}$ is a collection of closed halfspaces A, such that $\mathcal{C} = \{x \in \mathbb{R}^n | Ax \leq 0\}$.

Cones from vectors/rays and hyperplanes





Theorem (Weyl–Minkowski Theorem)

A convex polyhedral cone has both a vertex/ray and hyperplane description, which are equivalent.

Where Cones Commonly Show Up

- Solvability of a general system of linear equations (Farka's lemma)
- Integer point enumeration, Ehrhart Theory
- Discrete optimization, linear programming, feasibility problems
- Computational Complexity

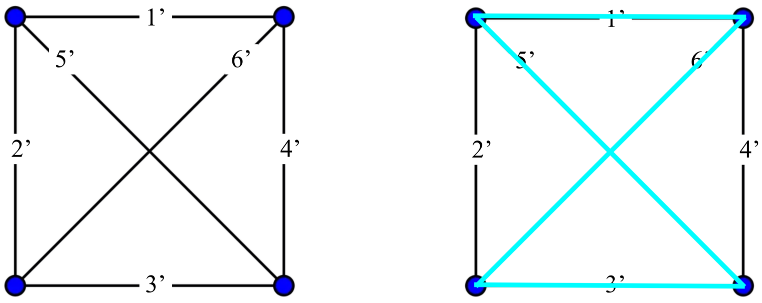
Where else might they show up?

Using Cones to Understand Graphs

Definition

- A *graph* $G = (V, E)$ is a set of vertices and edges.
- A *cycle of* G is a set of edges forming a path that returns to itself only once.

Example

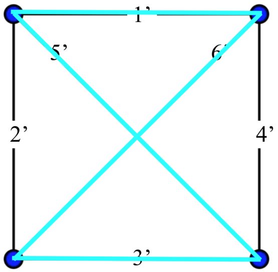


Using Cones to Understand Graphs

We can describe all the cycles of a graph using vectors!

- Let $c \in \{0,1\}^n$ be the indicator vector of a cycle of graph G, where $c_i = 1$ if $e_i \in E$ and 0 if not.

Example



Cycle in G = $(1, 0, 1, 0, 1, 1)$

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Example

$$C = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

- Every column is a cycle and rows are indexed by edges.

Using Cones to Understand Graphs

Using the set of cycles of G, we can generate the cone $\mathcal{C}_G$ over all cycles of G:

- $\mathcal{C}_G = \{\lambda_1 c_1 + \cdots + \lambda_n c_n \mid \lambda \in \mathbb{R}^n\}$

Example

$$\mathcal{C}_G = \left\{ \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \cdots + \lambda_7 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \middle| \lambda_i \in \mathbb{R}^7 \right\}$$

Using Cones to Understand Graphs

Why use cones? For a new perspective!

- **CDC conjecture:** For any graph G, there exists a set of cycles covering the edges of G so that every edge is in exactly 2 cycles.

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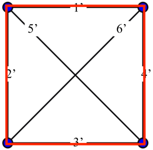

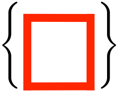

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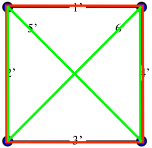

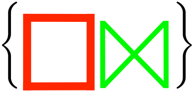

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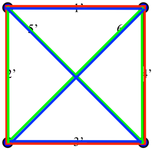

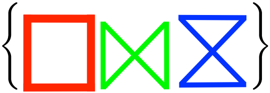

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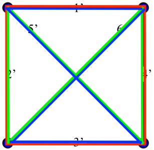

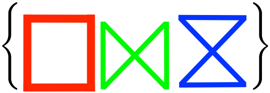

Via Cones: *The integral cone of cycles of G always contains $(2, 2, \ldots, 2)$.*

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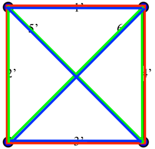

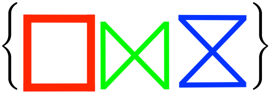

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- **In general:** Given vector $u = (u_1, u_2, \ldots, u_n)$, is there a set of cycles so that edge i is covered u_i many times?

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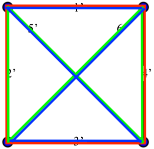

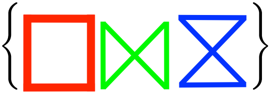

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Does the integral cone of cycles of G contain u?

References

- Beck and Robins, *Computing the Continuous Discretely*, Springer, 2015.
- De Loera, Hemmecke, and Köppe, *Algebraic and Geometric Ideas in the Theory of Discrete Optimization*, Society for Industrial and Applied Mathematics, 2012.
- Ziegler, *Lectures on Polytopes*, Springer, 1995.

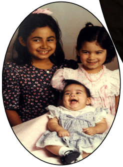
About me

SRJC

Instinct

SAN FRANCISCO STATE UNIVERSITY

Play time!



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Thank you!


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